

Stability of accelerating cosmology in two scalar-tensor theory: Little Rip versus de Sitter

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We develop the general reconstruction scheme in two scalar model. The quintom-like theory which may describe (different) non-singular Little Rip or de Sitter cosmology is reconstructed. (In)stability of such dark energy cosmologies as well as the flow to fixed points is studied. The stability of Little Rip universe which leads to dissolution of bound objects sometime in future indicates that no classical transition to de Sitter space occurs.

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I. INTRODUCTION

The observational data clearly indicate that current universe experiences the cosmic acceleration (dark energy epoch). We are still far away from complete understanding of dark energy which is often associated with some fluid with equation of state parameter w being close to -1 , or with modified gravity. The observational data favor the Λ CDM model whose equation of state parameter w is equal to -1 exactly. Nevertheless, it is quite possible that the universe evolution is governed by phantom ($w < -1$) or quintessence ($-1/3 > w > -1$) dark energy. If so then the future universe can evolve to finite-time singularity like Big Rip [1] or quintessence-related soft singularity of one of three types according to classification [2]. The natural prescription to cure finite-time future singularity may be found in frames of number of viable models of modified gravity [3]. However, it is still very interesting to understand if the solution of singularity problem may be found within fluid dark energy.

Recently, a new scenario to avoid future singularity has been proposed in [4] (for further development see Refs. [5–9]). According to this scenario the universe equation of state parameter w is less than -1 , so that dark energy density increases with time but w sufficiently rapidly approaches -1 asymptotically. In this way, the finite-time singularity is avoided. However, such proposed non-singular cosmology leads to dissolution of bound objects some when in future, similarly to Big Rip singularity. That is why the scenario was called Little Rip cosmology. The scalar models to describe Little Rip were introduced in Ref. [5], they turn out to be phantom-like scalars. It is known that one scalar models are stable in phantom phase and instable in non-phantom phase. Moreover, the large instability occurs when crossing the phantom-divide (or cosmological constant border). In order to understand Little Rip cosmology better, as well as its relation with asymptotically de Sitter universe and possible transitions between these two spaces, the more realistic description of Little Rip may be necessary.

In the present paper we develop such description in terms of two-scalar tensor theory which represents kind of quintom model [10, 11] (for review, see [12]). Indeed, one of two scalars is taken to be phantom.

In Section II, we consider a general formulation of reconstruction in two scalar model and investigate the stability of the solution. In this formulation, we construct a model which has a stable cosmological solution describing the phantom-divide crossing. In Section III, we reconstruct a model which describes the cosmological solutions with and without Little Rip and investigate the (in)stability of the solutions. The existence of the solution describing de Sitter space-time and the stability of the de Sitter solution when it exists as well as possible transition of Little Rip cosmology to de Sitter one are investigated. In Section IV, we also consider the reconstruction of the two scalar model in terms of the e-foldings N and investigate the flow of the solution in terms of dimensionless variables, which give the fixed points for some solutions. Some summary and outlook are given in Discussion section.

II. RECONSTRUCTION OF TWO SCALAR MODEL AND (IN)STABILITY

For the model with one scalar, the solution is stable in the phantom phase but unstable in the non-phantom phase. The instability becomes very large when crossing the cosmological constant line $w = -1$. In order to avoid this

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problem, one may consider two scalar model [13, 14].

We now consider the following two scalar model

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \omega(\phi) \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \eta(\chi) \partial_\mu \chi \partial^\mu \chi - V(\phi, \chi) \right\}. \quad (1)$$

Here $\omega(\phi)$ and $\eta(\chi)$ are functions of the scalar field ϕ and χ , respectively. The FRW equations give

$$\omega(\phi) \dot{\phi}^2 + \eta(\chi) \dot{\chi}^2 = -\frac{2}{\kappa^2} \dot{H}, \quad V(\phi, \chi) = \frac{1}{\kappa^2} (3H^2 + \dot{H}). \quad (2)$$

Then if

$$\omega(t) + \eta(t) = -\frac{2}{\kappa^2} f'(t), \quad V(t, t) = \frac{1}{\kappa^2} (3f(t)^2 + f'(t)), \quad (3)$$

the explicit solution follows

$$\phi = \chi = t, \quad H = f(t). \quad (4)$$

One may choose that ω should be always positive and η be always negative, for example

$$\begin{aligned} \omega(\phi) &= -\frac{2}{\kappa^2} \left\{ f'(\phi) - \sqrt{\alpha(\phi)^2 + f'(\phi)^2} \right\} > 0, \\ \eta(\chi) &= -\frac{2}{\kappa^2} \sqrt{\alpha(\chi)^2 + f'(\chi)^2} < 0. \end{aligned} \quad (5)$$

Here α is an arbitrary real function. We now define a new function $\tilde{f}(\phi, \chi)$ by

$$\tilde{f}(\phi, \chi) \equiv -\frac{\kappa^2}{2} \left(\int d\phi \omega(\phi) + \int d\chi \eta(\chi) \right), \quad (6)$$

which gives

$$\tilde{f}(t, t) = f(t). \quad (7)$$

If $V(\phi, \chi)$ is given by using $\tilde{f}(\phi, \chi)$ as

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left(3\tilde{f}(\phi, \chi)^2 + \frac{\partial \tilde{f}(\phi, \chi)}{\partial \phi} + \frac{\partial \tilde{f}(\phi, \chi)}{\partial \chi} \right), \quad (8)$$

the FRW and the following scalar field equations are also satisfied:

$$\begin{aligned} 0 &= \omega(\phi) \ddot{\phi} + \frac{1}{2} \omega'(\phi) \dot{\phi}^2 + 3H\omega(\phi) \dot{\phi} + \frac{\partial \tilde{V}(\phi, \chi)}{\partial \phi}, \\ 0 &= \eta(\chi) \ddot{\chi} + \frac{1}{2} \eta'(\chi) \dot{\chi}^2 + 3H\eta(\chi) \dot{\chi} + \frac{\partial \tilde{V}(\phi, \chi)}{\partial \chi}. \end{aligned} \quad (9)$$

In case of the one scalar model, the instability becomes infinite at the crossing $w = -1$ point. One may expect that such a divergence of the instability does not occur for model with two scalars.

By introducing the new quantities, X_ϕ , X_χ , and \tilde{Y} as

$$X_\phi \equiv \dot{\phi}, \quad X_\chi \equiv \dot{\chi}, \quad \tilde{Y} \equiv \frac{\tilde{f}(\phi, \chi)}{H}, \quad (10)$$

the FRW equations and the scalar field equations (9) are rewritten as

$$\begin{aligned} \frac{dX_\phi}{dN} &= -\frac{\omega'(\phi)}{2H\omega(\phi)} (X_\phi^2 - 1) - 3(X_\phi - \tilde{Y}), \\ \frac{dX_\chi}{dN} &= -\frac{\eta'(\chi)}{2H\eta(\chi)} (X_\chi^2 - 1) - 3(X_\chi - \tilde{Y}), \\ \frac{d\tilde{Y}}{dN} &= \frac{3X_\phi X_\chi (1 - \tilde{Y}^2)}{X_\phi + X_\chi} + \frac{\dot{H}}{H^2} \frac{X_\phi X_\chi + 1 - \tilde{Y} (X_\phi + X_\chi)}{X_\phi + X_\chi}. \end{aligned} \quad (11)$$

Here $d/dN \equiv H^{-1}d/dt$. In the solution (4), $X_\phi = X_\chi = \tilde{Y} = 1$. The following perturbation may be considered

$$X_\phi = 1 + \delta X_\phi, \quad X_\chi = 1 + \delta X_\chi, \quad \tilde{Y} = 1 + \delta \tilde{Y}. \quad (12)$$

Hence

$$\frac{d}{dN} \begin{pmatrix} \delta X_\phi \\ \delta X_\chi \\ \delta \tilde{Y} \end{pmatrix} = M \begin{pmatrix} \delta X_\phi \\ \delta X_\chi \\ \delta \tilde{Y} \end{pmatrix}, \quad M \equiv \begin{pmatrix} -\frac{\omega'(\phi)}{H\omega(\phi)} - 3 & 0 & 3 \\ 0 & -\frac{\eta'(\chi)}{H\eta(\chi)} - 3 & 3 \\ 0 & 0 & -3 - \frac{\ddot{H}}{H^2} \end{pmatrix}. \quad (13)$$

The eigenvalues of the matrix M are given by

$$M_\phi \equiv -\frac{\omega'(\phi)}{H\omega(\phi)} - 3, \quad M_\chi \equiv -\frac{\eta'(\chi)}{H\eta(\chi)} - 3, \quad M_{\tilde{Y}} \equiv -3 - \frac{\ddot{H}}{H^2}. \quad (14)$$

The eigenvalues (15) for the two scalar model are clearly finite. Hence, the instability, if any, could be finite and by choosing α in (5) properly, the instability can be removed, in general. In fact, right on the transition point where $\dot{H} = f'(t) = 0$ and therefore $f'(\phi) = f'(\chi) = 0$, for the choice in (5) with constant α , $\alpha(\phi) = \alpha(\chi) = \alpha > 0$, we find

$$\omega(\phi) = -\eta(\chi) = \frac{2\alpha}{\kappa^2}, \quad \omega'(\phi) = -\frac{2\ddot{H}}{\kappa^2}, \quad \eta'(\chi) = 0. \quad (15)$$

Then the eigenvalues (14) reduce to

$$M_\phi = \frac{\ddot{H}}{\alpha H} - 3, \quad M_\chi = M_{\tilde{Y}} = -3. \quad (16)$$

Then as long as $\frac{\ddot{H}}{\alpha H} < 3$, all the eigenvalues are negative and therefore the solution (4) is stable.

Hence, we gave general formulation of reconstruction in two scalar model and investigated the stability of the solution. By using this formulation, we can construct a model which has a stable cosmological solution corresponding to the phantom-divide crossing.

III. RECONSTRUCTION OF LITTLE RIP COSMOLOGY

In this section, by using the formulation of the previous section, we construct a model which may generate a Little Rip cosmology [4].

As an example we consider the following Hubble rate

$$H = H_0 e^{\lambda t}, \quad (17)$$

which corresponds to the Little Rip. Here H_0 and λ are positive constants. Eq. (17) shows that there is no curvature singularity for finite t .

When we ignore the contribution from matter, the equation of state (EoS) parameter w of the dark energy can be expressed in terms of the Hubble rate H as

$$w = -1 - \frac{2\dot{H}}{3H^2}. \quad (18)$$

Then if $\dot{H} > 0$, $w < -1$. By using Eq. (18), one finds

$$w = -1 - \frac{2\lambda}{3H_0} e^{-\lambda t}, \quad (19)$$

and therefore $w < -1$ and $w \rightarrow -1$ when $t \rightarrow +\infty$, and w is always less than -1 when \dot{H} is positive. The parameter A in [4] corresponds to $2\lambda/\sqrt{3}$ in (17) and is bounded as $2.74 \times 10^{-3} \text{ Gyr}^{-1} \leq A \leq 9.67 \times 10^{-3} \text{ Gyr}^{-1}$ by the results of the Supernova Cosmology Project [16].

In the model (17), H is always finite but increases exponentially, what generates the strong inertial force. The inertial force becomes larger and larger and any bound object is ripped. This phenomenon is called a ‘‘Little Rip’’ [4].

By choosing α in (5) as,

$$\alpha(t) = \alpha_0 e^{\lambda t}, \quad (20)$$

with a constant α_0 , we find $\omega(\phi)$ and $\eta(\chi)$ in (5) as follows

$$\omega(\phi) = \frac{2}{\kappa^2} \left(\sqrt{\alpha_0^2 + \lambda^2 H_0^2} - \lambda H_0 \right) e^{\lambda \phi}, \quad \eta(\chi) = -\frac{2}{\kappa^2} \sqrt{\alpha_0^2 + \lambda^2 H_0^2} e^{\lambda \chi}. \quad (21)$$

Using (6), we obtain

$$\tilde{f}(\phi, \chi) = \frac{1}{\lambda} \left\{ - \left(\sqrt{\alpha_0^2 + \lambda^2 H_0^2} - \lambda H_0 \right) e^{\lambda \phi} + \sqrt{\alpha_0^2 + \lambda^2 H_0^2} e^{\lambda \chi} \right\}, \quad (22)$$

and the potential in (8) is given by

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left\{ \frac{3}{\lambda^2} \left\{ - \left(\sqrt{\alpha_0^2 + \lambda^2 H_0^2} - \lambda H_0 \right) e^{\lambda \phi} + \sqrt{\alpha_0^2 + \lambda^2 H_0^2} e^{\lambda \chi} \right\}^2 - \left(\sqrt{\alpha_0^2 + \lambda^2 H_0^2} - \lambda H_0 \right) e^{\lambda \phi} + \sqrt{\alpha_0^2 + \lambda^2 H_0^2} e^{\lambda \chi} \right\}, \quad (23)$$

As written after (19), the parameter λ is bounded as $2.74 \times 10^{-3} \text{ Gyr}^{-1} \leq 2\lambda/\sqrt{3} \leq 9.67 \times 10^{-3} \text{ Gyr}^{-1}$ by the results of the Supernova Cosmology Project [16]. If we choose time so that the present universe corresponds to $t = 0$, we have $H_0 \sim 70 \text{ km/s Mpc}$.

For the model (17) with (20), the eigenvalues (14) are given by corresponding expressions when $\lambda t \gg 1$,

$$M_\phi = M_\chi = M_{\tilde{Y}} = -3 - \frac{\lambda}{H_0} e^{-\lambda t}, \quad (24)$$

which are negative. Therefore the solution is stable.

Let us consider the possibility that the universe could evolve to the de Sitter space-time. In order for the solution corresponding to the de Sitter space-time to exist, there should be an extremum in the potential and the potential should be positive there. If the extremum is local minimum with respect to ϕ , which is the canonical scalar and local maximum with respect to χ , which is non-canonical or phantom scalar, the solution is stable. For the potential (23), there is an extremum when

$$- \left(\sqrt{\alpha_0^2 + \lambda^2 H_0^2} - \lambda H_0 \right) e^{\lambda \phi} + \sqrt{\alpha_0^2 + \lambda^2 H_0^2} e^{\lambda \chi} = -\frac{\lambda^2}{6}, \quad (25)$$

where the value of the potential $V(\phi, \chi)$ is given by

$$V(\phi, \chi) = -\frac{\lambda^2}{12\kappa^2}, \quad (26)$$

which is negative and therefore there does not exist the solution corresponding to the de Sitter space-time. Hence, the universe does not evolve into the de Sitter space-time.

We now show that the (asymptotically) Little Rip solution (17) is always asymptotically stable. For large t , one assumes the solution behaves as (17). In three eigenvalues (14), for the asymptotically Little Rip solution which is in the phantom phase $\dot{H} > 0$, the eigenvalue $M_{\tilde{Y}}$ is negative. If we write $\alpha(\chi)$ in (5) as

$$\alpha(\chi) = \lambda H_0 q(\chi) e^{\lambda t}, \quad (27)$$

the eigenvalue M_χ (14) can be expressed as a function of $t = \chi$ as follows,

$$M_\chi = -\frac{q(t)(q'(t) + \lambda q(t)) + \lambda}{H_0 e^{\lambda t} (q(t)^2 + 1)} - 3. \quad (28)$$

If M_χ could be positive, $q(t)q'(t)$ must be negative. Since $q(t)^2$ is positive, $q(t)^2$ goes to a constant $q(t)^2 \rightarrow Q_0 \geq 0$. Then due to the factor $e^{\lambda t}$ in the denominator of the first term in (28), the first term goes to small value for large t and we find $M_\chi \rightarrow -3 < 0$. Therefore M_χ is asymptotically negative. For the eigenvalue M_ϕ , one gets

$$M_\phi = -\frac{\lambda + \frac{q(t)q'(t)}{\sqrt{q(t)^2 + 1}(\sqrt{q(t)^2 + 1} - 1)}}{H_0 e^{\lambda t}} - 3. \quad (29)$$

Then again in order that M_ϕ could be positive, $q(t)q'(t)$ must be negative and therefore $q(t)^2$ goes to a constant $q(t)^2 \rightarrow Q_0 \geq 0$. Due to the factor $e^{\lambda t}$ in the denominator of the first term in (29), the first term goes to small value for large t and $M_\phi \rightarrow -3 < 0$. Therefore, all the eigenvalues are negative and the Little Rip solution is asymptotically stable.

As another example, one can consider the following the model:

$$H = H_0 - H_1 e^{-\lambda t}. \quad (30)$$

Here H_0 , H_1 , and λ are positive constants and we assume $H_0 > H_1$ and $t > 0$. Since the second term decreases when t increases, the universe goes to asymptotically de Sitter space-time. Then from Eq. (18), we find

$$w = -1 - \frac{2\lambda H_1 e^{-\lambda t}}{3(H_0 - H_1 e^{-\lambda t})^2}. \quad (31)$$

As in the previous example (17), $w < -1$ and $w \rightarrow -1$ when $t \rightarrow +\infty$. In this model, there does not occur the Little Rip. The inertial force generating the Little Rip is given by

$$F_{\text{iner}} = m\ddot{a}/a = ml(\dot{H} + H^2). \quad (32)$$

Here we consider two points separated by a distance l and assume there is a particle with mass m at each of the points, Since the magnitudes of H and \dot{H} are bounded in the model (30), the Little Rip does not occur although the magnitudes of H and \dot{H} become larger and larger in the model (17).

For $t \rightarrow \infty$, Eq. (31) gives the asymptotic behavior of w to be

$$w \sim -1 - \frac{2\lambda H_1 e^{-\lambda t}}{3H_0^2}, \quad (33)$$

which is identical with (19) if we replace $\lambda H_1/H_0$ with λ .

By choosing α in (5) as,

$$\alpha(t) = \alpha_0 e^{-\lambda t}, \quad (34)$$

with a constant α_0 , we find $\omega(\phi)$ and $\eta(\chi)$ in (5) as follows

$$\omega(\phi) = \frac{2}{\kappa^2} \left(\sqrt{\alpha_0^2 + \lambda^2 H_1^2} - \lambda H_1 \right) e^{-\lambda \phi}, \quad \eta(\chi) = -\frac{2}{\kappa^2} \sqrt{\alpha_0^2 + \lambda^2 H_1^2} e^{-\lambda \chi}. \quad (35)$$

Using (6), one gets

$$\tilde{f}(\phi, \chi) = H_0 - \frac{1}{\lambda} \left\{ - \left(\sqrt{\alpha_0^2 + \lambda^2 H_1^2} - \lambda H_1 \right) e^{-\lambda \phi} + \sqrt{\alpha_0^2 + \lambda^2 H_1^2} e^{-\lambda \chi} \right\}, \quad (36)$$

and the potential (8) is given by

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left\{ \frac{3}{\lambda^2} \left\{ H_0 \lambda + \left(\sqrt{\alpha_0^2 + \lambda^2 H_1^2} - \lambda H_1 \right) e^{-\lambda \phi} - \sqrt{\alpha_0^2 + \lambda^2 H_1^2} e^{-\lambda \chi} \right\}^2 - \left(\sqrt{\alpha_0^2 + \lambda^2 H_1^2} - \lambda H_1 \right) e^{-\lambda \phi} + \sqrt{\alpha_0^2 + \lambda^2 H_1^2} e^{-\lambda \chi} \right\}, \quad (37)$$

For the model (30) with (34), the eigenvalues (14) are given by

$$M_\phi = M_\chi = -3 + \frac{\lambda}{H}, \quad M_{\tilde{Y}} = -3 - \frac{H_1 \lambda e^{-\lambda t}}{H^2}. \quad (38)$$

Therefore as long as $3 > \frac{\lambda}{H_0}$, the solution (30) is stable.

Since the solution can be unstable if $3 < \frac{\lambda}{H_0}$, we again consider the possibility that the universe could evolve to the de Sitter space-time. For the potential (37), there is an extremum when

$$- \left(\sqrt{\alpha_0^2 + \lambda^2 H_1^2} - \lambda H_1 \right) e^{\lambda \phi} + \sqrt{\alpha_0^2 + \lambda^2 H_1^2} e^{\lambda \chi} = -\frac{\lambda^2}{6} + \lambda H_0, \quad (39)$$

where the value of the potential $V(\phi, \chi)$ is given by

$$V(\phi, \chi) = \left(-\frac{\lambda^2}{12} + \lambda H_0 \right). \quad (40)$$

If $12 > \frac{\lambda}{H_0}$, which is consistent with the condition $3 < \frac{\lambda}{H_0}$ that the solution (30) is stable, $V(\phi, \chi)$ is positive and there is a solution corresponding to the de Sitter space-time. Therefore there is a possibility that the universe could evolve into the de Sitter space-time. Note, however, the solution (39) corresponds to the minimum with respect to both of ϕ and χ and therefore the de Sitter solution is not stable.

We may also consider the following the model:

$$H = H_0 + H_1 e^{-\lambda t}. \quad (41)$$

Here H_0 , H_1 , and λ are positive constants and we assume $H_0 > H_1$ and $t > 0$. Since the second term decreases when t increases, the universe goes to asymptotically de Sitter space-time. Then from Eq. (18), it follows

$$w = -1 + \frac{2\lambda H_1 e^{-\lambda t}}{3(H_0 - H_1 e^{-\lambda t})^2}. \quad (42)$$

Hence, the EoS parameter is always larger than -1 and $w \rightarrow -1$ when $t \rightarrow +\infty$. Therefore the universe is in non-phantom phase. By choosing α in (5) as in (34), we find $\omega(\phi)$ and $\eta(\chi)$ (5) as follows

$$\omega(\phi) = \frac{2}{\kappa^2} \left(\sqrt{\alpha_0^2 + \lambda^2 H_1^2} + \lambda H_1 \right) e^{-\lambda \phi}, \quad \eta(\chi) = -\frac{2}{\kappa^2} \sqrt{\alpha_0^2 + \lambda^2 H_1^2} e^{-\lambda \chi}. \quad (43)$$

Using (6), one gets

$$\tilde{f}(\phi, \chi) = H_0 - \frac{1}{\lambda} \left\{ - \left(\sqrt{\alpha_0^2 + \lambda^2 H_1^2} + \lambda H_1 \right) e^{-\lambda \phi} + \sqrt{\alpha_0^2 + \lambda^2 H_1^2} e^{-\lambda \chi} \right\}, \quad (44)$$

and the potential (8) is given by

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left\{ \frac{3}{\lambda^2} \left\{ H_0 \lambda + \left(\sqrt{\alpha_0^2 + \lambda^2 H_1^2} + \lambda H_1 \right) e^{-\lambda \phi} - \sqrt{\alpha_0^2 + \lambda^2 H_1^2} e^{-\lambda \chi} \right\}^2 - \left(\sqrt{\alpha_0^2 + \lambda^2 H_1^2} + \lambda H_1 \right) e^{-\lambda \phi} + \sqrt{\alpha_0^2 + \lambda^2 H_1^2} e^{-\lambda \chi} \right\}, \quad (45)$$

For the model (41) with (34), the eigenvalues (14) are given by

$$M_\phi = M_\chi = -3 + \frac{\lambda}{H}, \quad M_{\tilde{Y}} = -3 + \frac{H_1 \lambda e^{-\lambda t}}{H^2}. \quad (46)$$

If $3 > \frac{\lambda}{H}$ and $3 > \frac{H_1 \lambda e^{-\lambda t}}{H^2}$, the solution is stable.

Since the solution is unstable if $3 < \frac{\lambda}{H}$ or $3 < \frac{H_1 \lambda e^{-\lambda t}}{H^2}$, there is a possibility that the universe could evolve to the de Sitter space-time as in the model (37). For the potential (45), there is an extremum when

$$- \left(\sqrt{\alpha_0^2 + \lambda^2 H_1^2} + \lambda H_1 \right) e^{\lambda \phi} + \sqrt{\alpha_0^2 + \lambda^2 H_1^2} e^{\lambda \chi} = -\frac{\lambda^2}{6} + \lambda H_0, \quad (47)$$

where the value of the potential $V(\phi, \chi)$ is identical with that in (40) and positive if $12H_0 > \lambda$. The solution (47) is not stable, again.

As one more example, we consider the realistic model which contains the inflation at $t \rightarrow -\infty$, phantom crossing at $t = 0$, and the Little Rip when $t \rightarrow \infty$:

$$H = H_0 \cosh \lambda t. \quad (48)$$

Here H_0 and λ are positive constants. Since

$$\dot{H} = H_0 \lambda \sinh \lambda t, \quad (49)$$

we find $\dot{H} < 0$ when $t < 0$, that is, the universe is in non-phantom phase and $\dot{H} > 0$ when $t > 0$, that is, the universe is in phantom phase. There occurs the phantom crossing at $t = 0$. Therefore the present universe corresponds to $t \sim 0$. When $\lambda t \gg 1$, we find that the Hubble rate H behaves as

$$H \sim \frac{H_0}{2} e^{\lambda t}, \quad (50)$$

and therefore there occurs the Little Rip. The EoS parameter w is now given by

$$w = -1 - \frac{2\lambda \sinh \lambda t}{2H_0 \cosh^2 \lambda t}. \quad (51)$$

Hence, $w < -1$ when $t > 0$ and $w > -1$ when $t < 0$. In the limit $t \rightarrow \pm\infty$, $w \rightarrow -1$. Thus when $t \rightarrow -\infty$, there occurs the accelerating expansion, which may correspond to the inflation in the early universe. When $w = -\frac{1}{3}$, that is,

$$\frac{\lambda \sinh \lambda t}{H_0 \cosh^2 \lambda t} = -1, \quad (52)$$

there occurs the transition between non-accelerating expansion and accelerating expansion. There are two negative solutions in (52) in general. Let us denote the solution as t_i and t_l and assume $t_i < t_l < 0$. Then $t = t_i$ corresponds to the end of inflation and $t = t_l$ to the transition from the non-accelerating expansion to the late accelerating expansion in the present universe. More explicitly

$$\sinh \lambda t_i = -\frac{\lambda}{2H_0} - \sqrt{\left(\frac{\lambda}{2H_0}\right)^2 - 1}, \quad \sinh \lambda t_l = -\frac{\lambda}{2H_0} + \sqrt{\left(\frac{\lambda}{2H_0}\right)^2 - 1}. \quad (53)$$

Let the present universe corresponds to $t = t_{\text{present}}$. Since $t_i - t_{\text{present}} = 137 \times 10^8 \sim \frac{1}{H_{\text{present}}} = 138 \times 10^8$ years ($H_{\text{present}} \sim 70 \text{ km/s Mpc}$), if we assume $t_{\text{present}} = 0$, however, Eq. (52) does not have a solution for t_i . Then we may assume $t_{\text{present}} > 0$, that is, the present universe is after the phantom crossing. In this case in principle, one can solve (52) with respect to λ . Then we can obtain the value of t_l from the second equation in (53). Roughly one can expect the magnitude of the value could be $t_l \sim 50 \times 10^8$ years and therefore the realistic cosmology follows.

The acceleration by the gravitational force between the sun and the earth is given by

$$a_g = l\omega_A^2. \quad (54)$$

Here l is the distance between the sun and the earth and ω_A is the angular speed

$$\omega_A = \frac{2\pi}{1 \text{ year}} = 1.99 \times 10^{-6} \text{ s}^{-1}. \quad (55)$$

If the acceleration a_e of the inertial force by the expansion (32) exceeds a_g , there occurs the rip between the earth and the sun, that is

$$a_e = l(\dot{H} + H^2) \sim lH^2 \sim \frac{H_0^2}{4} e^{2\lambda t} > a_g. \quad (56)$$

which tells $e^{2\lambda t} = 8.35 \times 10^{19}$ or

$$\lambda t = 22.9. \quad (57)$$

If $\lambda = \mathcal{O}(10^{-10}) \text{ years}^{-1}$, $t \sim 10^{11}$ years.

For the model (48), by choosing α in (5) to be a constant $\alpha = H_0\lambda$, one finds

$$\omega(\phi) = \frac{2H_0\lambda}{\kappa^2} e^{-\lambda\phi}, \quad \eta(\chi) = -\frac{2H_0\lambda}{\kappa^2} \cosh \lambda\chi. \quad (58)$$

Using (6), it follows

$$\tilde{f}(\phi, \chi) = H_0 (e^{-\lambda\phi} + \sinh \lambda\chi), \quad (59)$$

TABLE I: The (in)stability of the solutions in the models

Models	Stability of the reconstructed solution	Existence of de Sitter solution	Stability of de Sitter solution
(21)	stable	no	—
(35)	stable if $3 > \frac{\lambda}{H_0}$	yes if $12 > \frac{\lambda}{H_0}$	unstable
(43)	stable if $3 > \frac{\lambda}{H}$ and $3 > \frac{H_1 \lambda e^{-\lambda t}}{H^2}$	yes if $12 > \frac{\lambda}{H_0}$	unstable
(58)	stable if $3 > \frac{\lambda}{H_0}$	no	—

and the potential in (8) is given by

$$V(\phi, \chi) = \frac{1}{\kappa^2} \left\{ H_0^2 (e^{-\lambda\phi} + \sinh \lambda\chi)^2 - H_0 \lambda e^{-\lambda\phi} + H_0 \lambda \cosh \lambda\chi \right\}. \quad (60)$$

For the model (50) with $\alpha = H_0 \lambda$, the eigenvalues (14) are given by

$$M_\phi = -3 + \frac{\lambda}{H_0 \cosh \lambda t}, \quad M_\chi = M_{\tilde{Y}} = -3 - \frac{\lambda \sinh \lambda t}{H_0 \cosh^2 \lambda t}. \quad (61)$$

Therefore if $3 > \frac{\lambda}{H_0}$, all the eigenvalues are negative and therefore the solution is stable.

For the potential (60), there is no extremum and therefore there does not exist the solution corresponding to the de Sitter space-time.

Thus, we constructed scalar models which describe the cosmological solutions with and without Little Rip and investigated the (in)stability of the solutions. We also investigated the existence of the solution describing de Sitter space-time and the stability of the de Sitter solution when it exists as well as possible transition of Little Rip cosmology to de Sitter one. The results are summarized in Table I.

IV. RECONSTRUCTION IN TERMS OF E-FOLDINGS AND SOLUTION FLOW

Here we consider the reconstruction of the two scalar model in terms of the e-foldings N . (For such a formulation in modified gravity, see [15].) We also investigate the flow of the solution by defining dimensionless variables, which give the fixed points for some solutions.

Let us consider two scalar model again. By using the e-foldings N the FRW equations (2) and the scalar field equations (9) are rewritten as

$$1 = \frac{\kappa^2}{6} \omega(\phi) \phi'^2 + \frac{\kappa^2}{6} \eta(\chi) \chi'^2 + \frac{\kappa^2 V(\phi, \chi)}{3H^2}, \quad (62)$$

$$1 + \frac{2H'}{3H} = -\frac{\kappa^2}{6} \omega(\phi) \phi'^2 - \frac{\kappa^2}{6} \eta(\chi) \chi'^2 + \frac{\kappa^2 V(\phi, \chi)}{3H^2}, \quad (63)$$

$$0 = \omega(\phi) \left[\phi'' + \left(3 + \frac{H'}{H} \right) \phi' \right] + \frac{1}{2} \omega_{,\phi}(\phi) \phi'^2 + \frac{V_{,\phi}(\phi, \chi)}{H^2}, \quad (64)$$

$$0 = \eta(\chi) \left[\chi'' + \left(3 + \frac{H'}{H} \right) \chi' \right] + \frac{1}{2} \eta_{,\chi}(\chi) \chi'^2 + \frac{V_{,\chi}(\phi, \chi)}{H^2}, \quad (65)$$

where ' denotes the derivative with respect to the e-foldings $N \equiv \ln a$. New function $\tilde{f}(\phi, \chi)$ is defined by

$$\tilde{f}(\phi, \chi) = f_0 \exp \left\{ -\frac{\kappa}{2} \left[\int d\phi \omega(\phi) + \int d\chi \eta(\chi) \right] \right\}. \quad (66)$$

Here f_0 is a dimensionless constant. We assume the potential $V(\phi, \chi)$ is given by

$$V(\phi, \chi) = \frac{3\tilde{f}(\phi, \chi)^2}{\kappa^4} \left[1 - \frac{\omega(\phi) + \eta(\chi)}{6} \right]. \quad (67)$$

Let

$$f(N) \equiv \tilde{f}\left(\frac{N}{\kappa}, \frac{N}{\kappa}\right). \quad (68)$$

Then if the functions $\omega(\phi)$ and $\eta(\chi)$ satisfy the following relations

$$\omega\left(\frac{N}{\kappa}\right) + \eta\left(\frac{N}{\kappa}\right) = -2\frac{f'(N)}{f(N)}, \quad (69)$$

a solution of ϕ , χ and H is given by

$$\phi = \chi = \frac{N}{\kappa}, \quad H = \frac{f(N)}{\kappa}. \quad (70)$$

Then one can obtain a model which reproduces arbitrary expansion history of the universe given by $H = f(N)/\kappa$, by choosing $\omega(\phi)$, $\eta(\chi)$ and $V(\phi, \chi)$ by (67) and (69).

Note that the solution (70) is one of the solutions in the model (1) with (67) and (69). In order to consider the structure of the space of the solutions, by defining dimensionless variables, we investigate the flow of the general solutions. Besides the solution (70), in general, there are other solutions including the one describing the de Sitter space corresponding to the extrema of the potential $V(\phi, \chi)$, where ϕ and χ are constant. We choose the variables so that both of the solution (70) and the de Sitter solution correspond to fixed points.

Let us introduce the dimensionless variables as follows:

$$X = \kappa\phi', \quad Y = \kappa\chi', \quad Z = \frac{\phi}{\chi}, \quad W = \kappa(\phi - \chi). \quad (71)$$

Eqs. (64) and (65) are rewritten as

$$\begin{aligned} X' = & 3(1 - X) + \frac{1}{2}\omega(\phi(Z, W))(X - X^2) + \frac{1}{2}\eta(\chi(Z, W))(X - Y^2) \\ & + \frac{\omega_{,\phi}(\phi(Z, W))}{2\kappa\omega(\phi(Z, W))} \frac{6(1 - X^2) - \eta(\chi(Z, W))(Y^2 - X^2)}{6 - \omega(\phi(Z, W)) - \eta(\chi(Z, W))}, \end{aligned} \quad (72)$$

$$\begin{aligned} Y' = & 3(1 - Y) + \frac{1}{2}\omega(\phi(Z, W))(Y - X^2) + \frac{1}{2}\eta(\chi(Z, W))(Y - Y^2) \\ & + \frac{\eta_{,\chi}(\chi(Z, W))}{2\kappa\eta(\chi(Z, W))} \frac{6(1 - Y^2) - \omega(\phi(Z, W))(X^2 - Y^2)}{6 - \omega(\phi(Z, W)) - \eta(\chi(Z, W))}, \end{aligned} \quad (73)$$

$$Z' = -\frac{(X - YZ)(1 - Z)}{W}, \quad (74)$$

$$W' = X - Y. \quad (75)$$

Now the Hubble rate H is given by

$$H = \frac{\tilde{f}(\phi(Z, W), \chi(Z, W))}{\kappa} \sqrt{\frac{6 - \omega(\phi(Z, W)) - \eta(\chi(Z, W))}{6 - \omega(\phi(Z, W))X^2 - \eta(\chi(Z, W))Y^2}}. \quad (76)$$

In order for the Hubble rate to be real, the values of X , Y , Z and W are restricted to a region

$$\frac{6 - \omega(\phi(Z, W))X^2 - \eta(\chi(Z, W))Y^2}{6 - \omega(\phi(Z, W)) - \eta(\chi(Z, W))} > 0. \quad (77)$$

When $\omega(\phi)$ and $\eta(\chi)$ satisfy (69), this system has two fixed points as follows:

Point A $(X, Y, Z, W) = (1, 1, 1, 0)$

Here the solution is given by (70).

Point B $(X, Y, Z, W) = (\beta_1, \beta_1, 1, 0)$

We now define $\beta(N)$ by

$$\beta(N) \equiv \beta_0 + \beta_1 N. \quad (78)$$

Here β_0 and β_1 are dimensionless constants which satisfy the identities

$$\frac{\omega_{,\phi}(\beta(N)/\kappa)}{\kappa\omega(\beta(N)/\kappa)} = \frac{\eta_{,\chi}(\beta(N)/\kappa)}{\kappa\eta(\beta(N)/\kappa)} = -\frac{6}{1+\beta_1} \left[1 + \frac{f'(\beta(N))}{3f(\beta(N))} \right] \left[1 - \frac{f'(\beta(N))}{3f(\beta(N))}\beta_1 \right]. \quad (79)$$

This point exists only if there exist β_0 and β_1 which satisfy (79). In this point, the solution is given by

$$\phi = \chi = \frac{\beta(N)}{\kappa}, \quad H = \frac{f(\beta(N))}{\kappa} \sqrt{\frac{3f(\beta(N)) + f'(\beta(N))}{3f(\beta(N)) + \beta_1^2 f'(\beta(N))}}. \quad (80)$$

Especially when $\beta_1 = 0$, this point describes de Sitter space-time.

We now choose $\omega(\phi)$ and $\eta(\chi)$ as

$$\omega(\phi) = -\frac{f'(\kappa\phi) - \sqrt{f'(\kappa\phi)^2 + \alpha(\kappa\phi)^2}}{f(\kappa\phi)} > 0, \quad (81)$$

$$\eta(\chi) = -\frac{f'(\kappa\chi) + \sqrt{f'(\kappa\chi)^2 + \alpha(\kappa\chi)^2}}{f(\kappa\chi)} < 0, \quad (82)$$

where $\alpha(N)$ is an arbitrary function. If we choose $\alpha(N) = \alpha_0 f'(N)$ and $f(N)$ being a monotonically increasing or decreasing function, $\omega(\phi)$ and $\eta(\chi)$ are respectively given by

$$\omega(\phi) = -\left(1 - \epsilon\sqrt{1 + \alpha_0^2}\right) \frac{f'(\kappa\phi)}{f(\kappa\phi)}, \quad \eta(\chi) = -\left(1 + \epsilon\sqrt{1 + \alpha_0^2}\right) \frac{f'(\kappa\chi)}{f(\kappa\chi)}. \quad (83)$$

Here α_0 is a dimensionless constant and $\epsilon \equiv f'(N)/|f'(N)|$. Then $V(\phi, \chi)$ has the following form:

$$V(\phi, \chi) = \frac{3}{\kappa^4} f(\kappa\phi)^{1-\epsilon\sqrt{1+\alpha_0^2}} f(\kappa\chi)^{1+\epsilon\sqrt{1+\alpha_0^2}} \times \left[1 + \frac{1-\epsilon\sqrt{1+\alpha_0^2}}{6} \frac{f'(\kappa\phi)}{f(\kappa\phi)} + \frac{1+\epsilon\sqrt{1+\alpha_0^2}}{6} \frac{f'(\kappa\chi)}{f(\kappa\chi)} \right]. \quad (84)$$

As an example, we consider

$$f(N) = f_0 e^{\lambda N}, \quad (85)$$

where λ is a dimensionless constant. Eq. (84) has a solution $H(N) = f_0 e^{\lambda N}/\kappa$. Then $\omega(\phi)$ and $\eta(\chi)$ are given by

$$\omega(\phi) = -\left(1 - \epsilon\sqrt{1 + \alpha_0^2}\right) \lambda, \quad \eta(\chi) = -\left(1 + \epsilon\sqrt{1 + \alpha_0^2}\right) \lambda, \quad (86)$$

and $V(\phi, \chi)$ is:

$$V(\phi, \chi) = \frac{3f_0^2}{\kappa^4} \left(1 + \frac{\lambda}{3}\right) \exp \left[\left(1 - \epsilon\sqrt{1 + \alpha_0^2}\right) \lambda \kappa \phi + \left(1 + \epsilon\sqrt{1 + \alpha_0^2}\right) \lambda \kappa \chi \right]. \quad (87)$$

The EoS parameter of the point A is

$$w = -1 - \frac{2}{3}\lambda, \quad (88)$$

which is independent of N . If $\lambda > 0$, the point B exists and is located in $(3/\lambda, 3/\lambda, 1, 0)$. Then the solution is given by

$$\phi = \chi = \frac{3(N - N_0)}{\lambda\kappa}, \quad H = \frac{f_0}{\kappa} \sqrt{\frac{\lambda}{3}} e^{3(N - N_0)}. \quad (89)$$

Here N_0 is an arbitrary constant. The EoS parameter of this point is -3 . In this model, the dynamics of X and Y are independent of Z and W . Therefore, we consider a small fluctuation from each fixed point by

$$X(N) = X_0 + \delta X(N), \quad Y(N) = Y_0 + \delta Y(N). \quad (90)$$

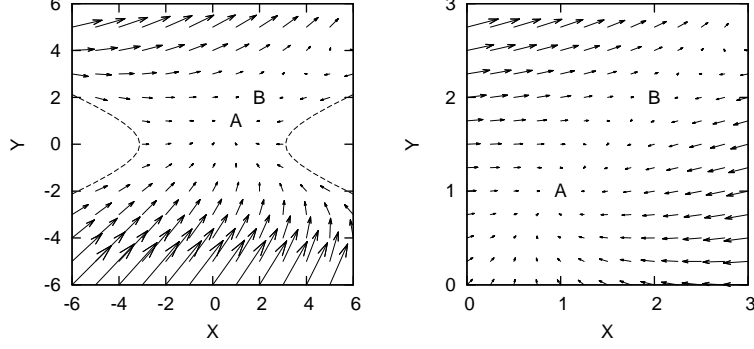


FIG. 1: Each vector denotes $(X'/50, Y'/50)$, which is independent of Z and W . The parameters are $\lambda = 3/2$ and $\alpha_0 = 1$. The point A is located in $(1, 1)$, where the EoS parameter is -2 . The point B is located in $(2, 2)$, where the EoS parameter is -3 .

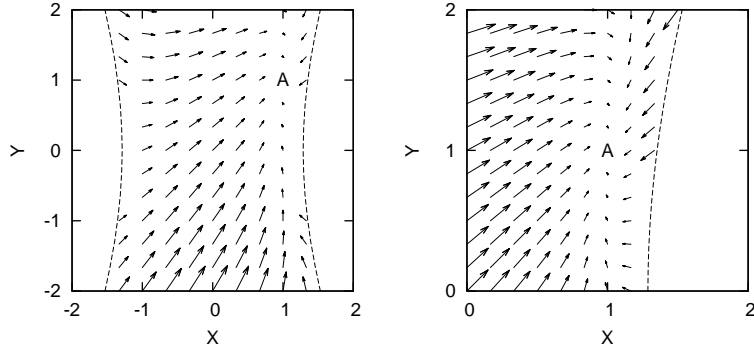


FIG. 2: Each vector denotes $(X'/20, Y'/20)$, which is independent of Z and W . The parameters are $\lambda = -3/2$ and $\alpha_0 = 1$. The point A is located in $(1, 1)$, where the EoS parameter is 0.

Here X_0 and Y_0 are the values of X and Y in each fixed point. Then (72) and (73) have the following form:

$$\begin{pmatrix} \delta X' \\ \delta Y' \end{pmatrix} = \begin{pmatrix} -3 - \lambda + (1 - \epsilon\sqrt{1 + \alpha_0^2}) \lambda X_0 & (1 + \epsilon\sqrt{1 + \alpha_0^2}) \lambda Y_0 \\ (1 - \epsilon\sqrt{1 + \alpha_0^2}) \lambda X_0 & -3 - \lambda + (1 + \epsilon\sqrt{1 + \alpha_0^2}) \lambda Y_0 \end{pmatrix} \begin{pmatrix} \delta X \\ \delta Y \end{pmatrix}. \quad (91)$$

The eigenvalues of this matrix (91) are given by

$$\sigma_1 = -3 - \lambda, \quad \sigma_2 = -3 - \lambda + \left(1 - \epsilon\sqrt{1 + \alpha_0^2}\right) \lambda X_0 + \left(1 + \epsilon\sqrt{1 + \alpha_0^2}\right) \lambda Y_0. \quad (92)$$

These indicate that the point A is stable if $-3 < \lambda < 3$ and unstable if $\lambda < -3$ or $\lambda > 3$. Similarly, the point B is stable if $\lambda > 3$ and unstable if $0 < \lambda < 3$. The dynamics of X and Y are shown in FIG. 1 and FIG. 2.

We now consider another example

$$f(N) = f_0 N^\gamma, \quad (93)$$

where γ is a dimensionless constant. We should note that the case $\gamma = 1$ corresponds to the Little Rip model (17). Then $\omega(\phi)$ and $\eta(\chi)$ are given by

$$\omega(\phi) = -\left(1 - \epsilon\sqrt{1 + \alpha_0^2}\right) \frac{\gamma}{\kappa\phi}, \quad \eta(\chi) = -\left(1 + \epsilon\sqrt{1 + \alpha_0^2}\right) \frac{\gamma}{\kappa\chi}, \quad (94)$$

and $V(\phi, \chi)$ has the following form:

$$V(\phi, \chi) = \frac{3f_0^2}{\kappa^4} (\kappa\phi)^{1 - \epsilon\sqrt{1 + \alpha_0^2}} (\kappa\chi)^{1 + \epsilon\sqrt{1 + \alpha_0^2}} \times \left[1 + \frac{1 - \epsilon\sqrt{1 + \alpha_0^2}}{6} \frac{\gamma}{\kappa\phi} + \frac{1 + \epsilon\sqrt{1 + \alpha_0^2}}{6} \frac{\gamma}{\kappa\chi} \right]. \quad (95)$$

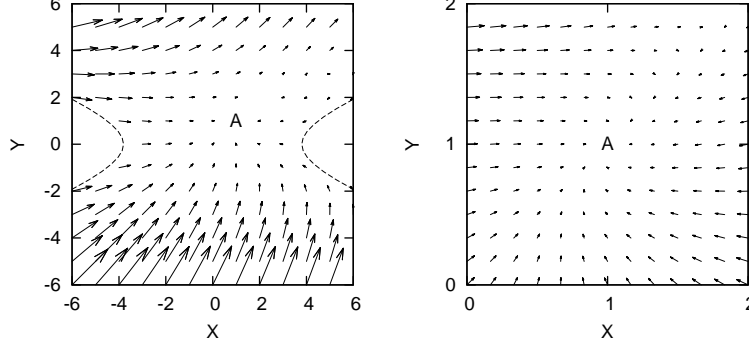


FIG. 3: Each vector denotes $(X'/50, Y'/50)$ with $Z = 1$ and $W = 0$ ($\phi = \chi = 1/\kappa$). The parameters are $\gamma = 1$ and $\alpha_0 = 1$. The point A is located in $(1, 1)$, which corresponds to the Little Rip universe.

The EoS parameter of the point A is

$$w = -1 - \frac{2\gamma}{3N}, \quad (96)$$

which becomes -1 when $N \rightarrow \infty$. If $\gamma < 1/2$, the point B exists and is located in $(0, 0, 1, 0)$. Then the solution is

$$\phi = \chi = \frac{1 - 2\gamma}{6\kappa}, \quad H = \frac{f_0}{\sqrt{6\kappa}} \left(\frac{6}{1 - 2\gamma} \right)^{1/2 - \gamma}. \quad (97)$$

This point corresponds to the de Sitter space-time. We consider a small fluctuation from the point B by

$$\phi(N) = \frac{1 - 2\gamma}{6\kappa} + \delta\phi(N), \quad \chi(N) = \frac{1 - 2\gamma}{6\kappa} + \delta\chi(N). \quad (98)$$

Then (72) and (73) have the following form:

$$0 = (1 - 2\gamma) \begin{pmatrix} \delta\phi'' \\ \delta\chi'' \end{pmatrix} + 3 \begin{pmatrix} \delta\phi' \\ \delta\chi' \end{pmatrix} - 18 \begin{pmatrix} 1 - (1 - \epsilon\sqrt{1 + \alpha_0^2})\gamma & -(1 + \epsilon\sqrt{1 + \alpha_0^2})\gamma \\ -(1 - \epsilon\sqrt{1 + \alpha_0^2})\gamma & 1 - (1 + \epsilon\sqrt{1 + \alpha_0^2})\gamma \end{pmatrix} \begin{pmatrix} \delta\phi \\ \delta\chi \end{pmatrix}. \quad (99)$$

The solution of this equation (99) is given by

$$\begin{pmatrix} \delta\phi \\ \delta\chi \end{pmatrix} = \begin{pmatrix} 1 + \epsilon\sqrt{1 + \alpha_0^2} \\ -1 + \epsilon\sqrt{1 + \alpha_0^2} \end{pmatrix} \frac{C_{1+} e^{\sigma_{1+}N} + C_{1-} e^{\sigma_{1-}N}}{\kappa} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{C_{2+} e^{\sigma_{2+}N} + C_{2-} e^{\sigma_{2-}N}}{\kappa}. \quad (100)$$

Here $C_{1\pm}$ and $C_{2\pm}$ are arbitrary constants and $\sigma_{1\pm}$ and $\sigma_{2\pm}$ are given by

$$\sigma_{1\pm} = \frac{-3 \pm 3\sqrt{1 + 8(1 - 2\gamma)}}{2(1 - 2\gamma)}, \quad \sigma_{2\pm} = \frac{-3 \pm 3\sqrt{1 + 8(1 - 2\gamma)^2}}{2(1 - 2\gamma)}. \quad (101)$$

These indicate that the point B is always unstable. On the other hand, when $\phi, \chi \rightarrow \infty$, the solution of X and Y are given by

$$X = 1 - X_1 e^{-3N}, \quad Y = 1 - Y_1 e^{-3N}. \quad (102)$$

Here X_1 and Y_1 are arbitrary constants. This indicates that the values of X and Y approach to $(1, 1)$ if $\phi, \chi \rightarrow \infty$ when $N \rightarrow \infty$. Then the Hubble rate approaches to $f_0 N^\gamma / \kappa$, which corresponds to the Little Rip universe if $\gamma = 1$. The dynamics of X and Y are shown in FIG. 3 and FIG. 4.

Then we have completed the formulation of the reconstruction in terms of the e-foldings. The e-foldings description is directly related with redshift and therefore, with the cosmological observations. We also investigated the flow for the solution, which shows the (in)stability of the reconstructed solution obtained in a large range. Even if the solution is stable, when the stable region is small, the evolution of the universe depends strongly on the initial conditions. If the

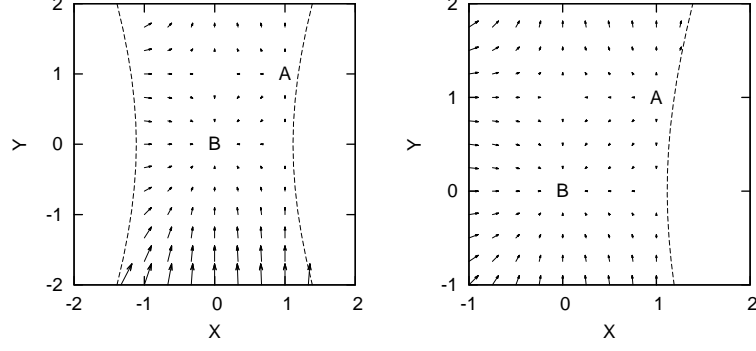


FIG. 4: Each vector denotes $(X'/20, Y'/20)$ with $Z = 1$ and $W = 0$ ($\phi = \chi = 1/2\kappa$). The parameters are $\gamma = -1$ and $\alpha_0 = 1$. The point A is located in $(1, 1)$. The point B is located in $(0, 0)$, which corresponds to the de Sitter universe.

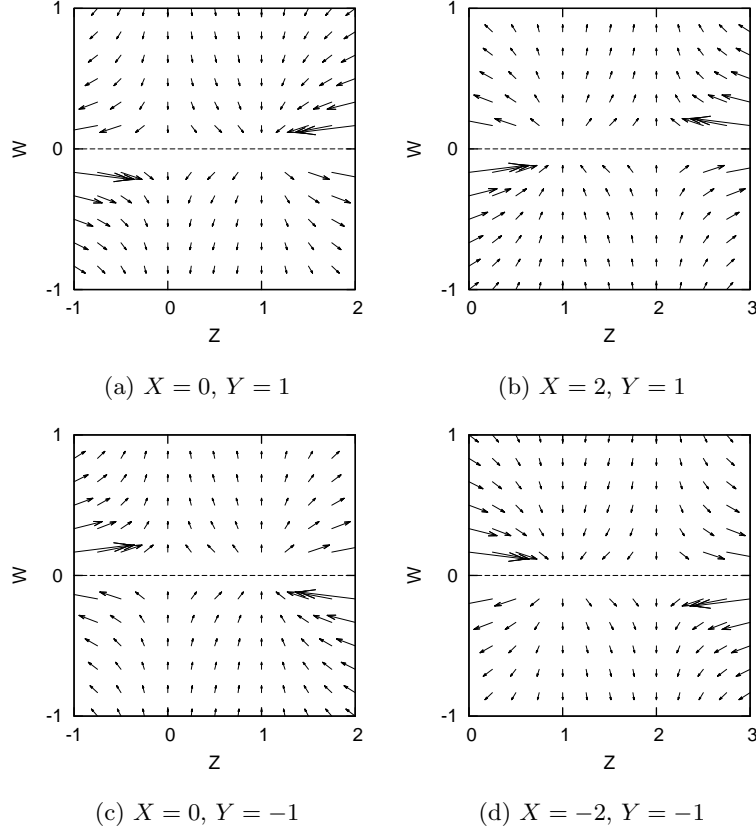


FIG. 5: Each vector denotes $(Z'/20, W'/20)$, which is independent of the form of $\omega(\phi)$, $\eta(\chi)$ and $V(\phi, \chi)$. The dynamics of Z and W are classified into four types according to the values of X and Y as (a) $0 < Y, X < Y$, (b) $0 < Y < X$, (c) $0 > Y, X > Y$ and (d) $0 > Y > X$. The fixed points are located in $(1, 0)$.

initial condition is out of the range, the universe does not always evolve to the solution obtained by the reconstruction. On the other hand, if the stable region is large enough, even if the universe started from an initial condition in a rather large region, the universe evolves to the solution obtained by the reconstruction.

As mentioned above, the model (93) with $\gamma = 1$ corresponds to the Little Rip model (17). In fact, Eq. (17) indicates that

$$N = \frac{H_0}{\lambda} e^{\lambda t}, \quad (103)$$

and therefore

$$H = \lambda N. \quad (104)$$

By comparing (104) with (93), we find $f_0 = \lambda$. Then as written after (19), the parameter f_0 is bounded as $2.74 \times 10^{-3} \text{ Gyr}^{-1} \leq 2f_0/\sqrt{3} \leq 9.67 \times 10^{-3} \text{ Gyr}^{-1}$ by the results of the Supernova Cosmology Project [16]. If we choose so that the present universe corresponds to $N = N_0$, we have $f_0 N_0 \sim 70 \text{ km/s Mpc}$.

V. DISCUSSION

In summary, we gave a general formulation of reconstruction in two scalar model and investigated the stability of the solution. This formulation helps us to construct a model which has a stable cosmological solution describing the phantom-divide crossing. By using the formulation, we constructed a model which describe the cosmological solutions with and without Little Rip and investigated the (in)stability of the solutions. The existence of the solution describing de Sitter space-time was also investigated and furthermore the stability of the de Sitter solution when it exists as well as possible transition of Little Rip cosmology to de Sitter one was investigated. We also considered the reconstruction of the two scalar model in terms of the e-foldings N and investigated the flow of the solution by defining dimensionless variables, which give the fixed points for some solutions.

Finally, let us make several remarks about the relation of the qualitative behavior of the Universe evolution and the shape of the scalar potential.

In case of the usual canonical scalar field as ϕ in (1), when the field climbs up the potential, the kinetic energy decreases until the kinetic energy vanishes. Even in case of the phantom field as χ in (1) with non-canonical kinetic term, the kinetic energy decreases when the field climbs up the potential. In case of the phantom field, the kinetic energy is unbounded below and therefore the absolute value of the kinetic energy increases when the field climbs up the potential. The big rip or Little Rip occurs when the potential goes to infinity. If the potential tends to infinity in the finite future, the evolution corresponds to the big rip but if the potential goes to infinity in the infinite future, the evolution corresponds to the Little Rip. Then the necessary condition that the big or Little Rip could occur is

1. The potential does not have maximum and it goes to infinity.
2. There is a path in the potential that the potential becomes infinite but the kinetic energy of the canonical scalar field is vanishing.

Since we identify the scalar field ϕ and χ with the cosmological time, the second condition means $\omega(\phi)$ in (1) goes to zero when ϕ goes to infinity and therefore the phantom field χ dominates. Conversely, if there is a maximum in the potential or there is no path in the potential that the potential becomes infinite but the kinetic energy of the canonical scalar field goes to zero, there does not occur big rip nor Little Rip.

Let us suppose the case that there is a maximum in the potential. If the fields stay near the potential maximum, the universe becomes asymptotically de Sitter space-time. If the fields go through the maximum and the potential decreases, the kinetic energy of the canonical scalar field increases but the absolute value of the kinetic energy of the phantom field decreases. If the kinetic energy of the phantom field goes to zero, the canonical field becomes dominant and the Universe could enter the non-phantom (quintessence) phase and there might occur the deceleration phase in future.

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